Indian Statistical Institute
Mid Semester Exam
Algebra-I
27-09-2010

Time : 3 hours   Max. Marks : 40

Answer all questions. All questions carry equal marks.

(1) Define the group $\mathcal{I}nn(G)$ of inner automorphisms of a group G. Show that $\mathcal{I}nn(G) \cong G/\mathcal{Z}(G)$, where $\mathcal{Z}(G)$ is the center of G.

(2) Describe the automorphism group of a cyclic group of order n.

(3) Let G have even order $2n$. Suppose that exactly half of the elements of G have order 2 and the rest form a subgroup H of order n. Prove that $o(H)$ is odd and that H is abelian.

(4) Let G be a group and let H be a subgroup of G. Let G act by left multiplication on the set G/H of all left cosets of H in G. Let ψ_H denote the associated permutation representation of this action. Show that

(a) The action of G on G/H is transitive.

(b) Stabiliser of the left coset $eH = H$ is the subgroup H.

(c) Kernel of ψ_H is the largest normal subgroup of G contained in H.

(5) Show that if G is a group of order p^n, where p is a prime and n is a positive integer, then every subgroup of G of index p is normal in G. (Hint: Use the action in problem 4).